

# Chapter 3

- Sequences
- Mathematical Induction
  - Recursion

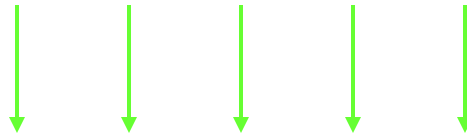
# • Sequences

# Sequences

- **Sequences** represent **ordered lists** of elements.
- A **sequence** is defined as a function from a subset of  $\mathbf{N}$  to a set  $S$ . We use the notation  $a_n$  to denote the image of the integer  $n$ . We call  $a_n$  a term of the sequence.

- **Example:**

- subset of  $\mathbf{N}$ :      1   2   3   4   5   ...



S:            2   4   6   8   10   ...

# Sequences

- We use the notation  $\{a_n\}$  to describe a sequence.
- **Important: Do not confuse this with the  $\{\}$  used in set notation.**
- It is convenient to describe a sequence with a **formula**.
- For example, the sequence on the previous slide can be specified as  $\{a_n\}$ , where  $a_n = 2n$ .

# The Formula Game

What are the formulas that describe the following sequences  $a_1, a_2, a_3, \dots$  ?

•  $1, 3, 5, 7, 9, \dots$

$$a_n = 2n - 1$$

$-1, 1, -1, 1, -1, \dots$

$$a_n = (-1)^n$$

$2, 5, 10, 17, 26, \dots$

$$a_n = n^2 + 1$$

$0.25, 0.5, 0.75, 1, 1.25, \dots$

$$a_n = 0.25n$$

$3, 9, 27, 81, 243, \dots$

$$a_n = 3^n$$

# Strings

- Finite sequences are also called **strings**, denoted by  $a_1a_2a_3\dots a_n$ .
- The **length** of a string  $S$  is the number of terms that it consists of.
- The **empty string** contains no terms at all. It has length zero.

# Summations

What does  $\sum_{j=m}^n a_j$  stand for?

- It represents the sum  $a_m + a_{m+1} + a_{m+2} + \dots + a_n$ .
- The variable  $j$  is called the **index of summation**, running from its **lower limit**  $m$  to its **upper limit**  $n$ . We could as well have used any other letter to denote this index.

# Summations

How can we express the sum of the first 1000 terms of the sequence  $\{a_n\}$  with  $a_n = n^2$  for  $n = 1, 2, 3, \dots$  ?

We write it as  $\sum_{j=1}^{1000} j^2$ .

What is the value of  $\sum_{j=1}^6 j$  ?

• It is  $1 + 2 + 3 + 4 + 5 + 6 = 21$ .

What is the value of  $\sum_{j=1}^{100} j$  ?

It is so much work to calculate this...



# Summations

• It is said that Friedrich Gauss came up with the following formula:

$$\sum_{j=1}^n j = \frac{n(n+1)}{2}$$

**When you have such a formula, the result of any summation can be calculated much more easily, for example:**

$$\sum_{j=1}^{100} j = \frac{100(100+1)}{2} = \frac{10100}{2} = 5050$$

# Arithmetic Series

•How does:  $\sum_{j=1}^n j = \frac{n(n+1)}{2}$  ???

Observe that:

$$1 + 2 + 3 + \dots + n/2 + (n/2 + 1) + \dots + (n - 2) + (n - 1) + n$$

$$= [1 + n] + [2 + (n - 1)] + [3 + (n - 2)] + \dots + [n/2 + (n/2 + 1)]$$

$$= (n + 1) + (n + 1) + (n + 1) + \dots + (n + 1) \quad (\text{with } n/2 \text{ terms})$$

$$= n(n + 1)/2.$$

# Geometric Series

•How does:  $\sum_{j=0}^n a^j = \frac{a^{(n+1)} - 1}{(a - 1)}$  ???

Observe that:

$$S = 1 + a + a^2 + a^3 + \dots + a^n$$

$$aS = a + a^2 + a^3 + \dots + a^n + a^{(n+1)}$$

$$\text{so, } (aS - S) = (a - 1)S = a^{(n+1)} - 1$$

Therefore,  $1 + a + a^2 + \dots + a^n = (a^{(n+1)} - 1) / (a - 1)$ .

For example:  $1 + 2 + 4 + 8 + \dots + 1024 = 2047$ .

# Useful Series

- 1.  $\sum_{j=1}^n j = \frac{n(n+1)}{2}$
- 2.  $\sum_{j=0}^n a^j = \frac{a^{(n+1)} - 1}{(a-1)}$
- 3.  $\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6}$
- 4.  $\sum_{j=1}^n j^3 = \frac{n^2(n+1)^2}{4}$

# Double Summations

•Corresponding to nested loops in C or Java, there is also double (or triple etc.) summation:

•Example:

$$\begin{aligned} & \sum_{i=1}^5 \sum_{j=1}^2 ij \\ &= \sum_{i=1}^5 (i + 2i) \\ &= \sum_{i=1}^5 3i \\ &= 3 + 6 + 9 + 12 + 15 = 45 \end{aligned}$$

Table 2 in Section 3.2 contains some very useful formulas for calculating sums.

Follow me for a walk through...

- **Mathematical**
  - **Induction**

# Induction

- The **principle of mathematical induction** is a useful tool for proving that a certain predicate is true for **all natural numbers**.
- It cannot be used to discover theorems, but only to prove them.