Chapter 3

- Sequences
- Mathematical Induction
- Recursion

- Sequences

Sequences

- **Sequences** represent **ordered lists** of elements.
- A **sequence** is defined as a function from a subset of **N** to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.
- **Example:**
- subset of **N**: 1 2 3 4 5 …

S: 2 4 6 8 10 …

Sequences

- We use the notation $\{a_n\}$ to describe a sequence.

- Important: Do not confuse this with the {} used in set notation.

- It is convenient to describe a sequence with a **formula**.

- For example, the sequence on the previous slide can be specified as $\{a_n\}$, where $a_n = 2n$.

The Formula Game

What are the formulas that describe the following sequences $a_1, a_2, a_3, \ldots$?

- 1, 3, 5, 7, 9, … $a_n = 2n - 1$

-1, 1, -1, 1, -1, … $a_n = (-1)^n$

2, 5, 10, 17, 26, … $a_n = n^2 + 1$

0.25, 0.5, 0.75, 1, 1.25 … $a_n = 0.25n$

3, 9, 27, 81, 243, … $a_n = 3^n$

Strings

- Finite sequences are also called **strings**, denoted by $a_1a_2a_3\ldots a_n$.

- The **length** of a string S is the number of terms that it consists of.

- The **empty string** contains no terms at all. It has length zero.

Summations

What does $\sum_{j=m}^{n} a_j$ stand for?

- It represents the sum $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$.

- The variable j is called the **index of summation**, running from its **lower limit** m to its **upper limit** n. We could as well have used any other letter to denote this index.

Summations

How can we express the sum of the first 1000 terms of the sequence $\{a_n\}$ with $a_n=n^2$ for $n = 1, 2, 3, \ldots$?

We write it as $\sum_{j=1}^{1000} j^2$.

What is the value of $\sum_{j=1}^{6} j$?

- It is 1 + 2 + 3 + 4 + 5 + 6 = 21.

What is the value of $\sum_{j=1}^{100} j$?

It is so much work to calculate this…

Summations

- It is said that Friedrich Gauss came up with the following formula:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

When you have such a formula, the result of any summation can be calculated much more easily, for example:

$$\sum_{j=1}^{100} j = \frac{100(100+1)}{2} = \frac{10100}{2} = 5050$$

Arithemetic Series

- How does: $$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$???

Observe that:

1 + 2 + 3 +…+ n/2 + (n/2 + 1) +…+ (n - 2) + (n - 1) + n

= [1 + n] + [2 + (n - 1)] + [3 + (n - 2)] +…+ [n/2 + (n/2 + 1)]

= (n + 1) + (n + 1) + (n + 1) + … + (n + 1) ***(with n/2 terms)***

= n(n + 1)/2.

Geometric Series

- How does: $$\sum_{j=0}^{n} a^j = \frac{a^{(n+1)} - 1}{(a-1)}$$???

Observe that:

$$S = 1 + a + a^2 + a^3 + \ldots + a^n$$

$$aS = a + a^2 + a^3 + \ldots + a^n + a^{(n+1)}$$

so, $(aS - S) = (a - 1)S = a^{(n+1)} - 1$

Therefore, $1 + a + a^2 + \ldots + a^n = (a^{(n+1)} - 1) / (a - 1)$.

For example: $1 + 2 + 4 + 8 + \ldots + 1024 = 2047$.

Useful Series

- 1. $$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$
- 2. $$\sum_{j=0}^{n} a^j = \frac{a^{(n+1)} - 1}{(a-1)}$$
- 3. $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$
- 4. $$\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$$

Double Summations

•Corresponding to nested loops in C or Java, there is also double (or triple etc.) summation:

•Example:

$$\sum_{i=1}^{5}\sum_{j=1}^{2} ij$$

$$= \sum_{i=1}^{5}(i+2i)$$

$$= \sum_{i=1}^{5} 3i$$

$$= 3+6+9+12+15 = 45$$

Table 2 in Section 3.2 contains some very useful formulas for calculating sums.

Follow me for a walk through...

- Mathematical
- Induction

Induction

- The **principle of mathematical induction** is a useful tool for proving that a certain predicate is true for **all natural numbers**.

- It cannot be used to discover theorems, but only to prove them.